

MAS212 Linear Algebra I Exercises 2 (with solutions)

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Assessed submission deadline 5pm on Friday 13th October 2006 in the blue box on the ground floor. Your submission must have your FULL NAME and STUDENT NUMBER PRINTED CLEARLY at the top and separate sheets must be stapled together. Marks out of 100 are shown in square brackets after each question number.

1. [30] In the following, *prove* that U either is or is not a vector subspace of the specified vector space.

- (a) Let $U = \{(x, y, z) \in \mathbb{R}^3 \mid x + y = 0\}$. Is U a vector subspace of $\mathbb{R}^3$?
- (b) Let $U = \{(x, y, z) \in \mathbb{R}^3 \mid x + y = 1\}$. Is U a vector subspace of $\mathbb{R}^3$?
- (c) Let $U = \{(w, x, y, z) \in \mathbb{R}^4 \mid w + x = 0, y + z = 0\}$. Is U a vector subspace of $\mathbb{R}^4$?
- (d) Let $U = \{(x, y) \in \mathbb{R}^2 \mid x = \pm y\}$. Is U a vector subspace of $\mathbb{R}^2$?
- (e) Let $U = \{(x, y) \in \mathbb{R}^2 \mid x^2 - y^2 = 1\}$. Is U a vector subspace of $\mathbb{R}^2$?
- (f) Let $U = \{(x, y) \in \mathbb{R}^2 \mid x = y = 0\}$. Is U a vector subspace of $\mathbb{R}^2$?

Solution

It is necessary to check the vector subspace conditions, in one form or another.

(a) Yes, because the constraint is homogeneous linear. $(0, 0, 0)$ satisfies the constraint since $0 + 0 = 0$, so U contains the zero vector. To check closure, let

$u = (x_1, y_1, z_1), v = (x_2, y_2, z_2) \in U$. Then $x_1 + y_1 = 0$ (1), $x_2 + y_2 = 0$ (2).

The vector $u + v = (x_1 + x_2, y_1 + y_2, z_1 + z_2) \in U$ if it satisfies the constraint on U that

$(x_1 + x_2) + (y_1 + y_2) = 0$. Adding constraints (1) and (2) gives $(x_1 + x_2) + (y_1 + y_2) = 0$.

Hence $u + v \in U$ and U is closed under addition. For any $a \in \mathbb{R}$, the vector

$au = (ax_1, ay_1, az_1) \in U$ if it satisfies the constraint on U that $(ax_1) + (ay_1) = 0$.

Multiplying constraint (1) by a gives $(ax_1) + (ay_1) = 0$. Hence $au \in U$ and U is closed under scalar multiplication. Therefore, U is a vector subspace of $\mathbb{R}^3$.

(b) No, because the constraint is not homogeneous linear. In particular, $(0, 0, 0) \notin U$ since $0 + 0 \neq 1$, so U does not contain the zero vector.

(c) Yes, because the constraints are homogeneous linear. $(0, 0, 0, 0)$ satisfies the constraints because $0 + 0 = 0, 0 + 0 = 0$, so U contains the zero vector. To check closure, let

$u = (w_1, x_1, y_1, z_1), v = (w_2, x_2, y_2, z_2) \in U$. Then

$w_1 + x_1 = 0, y_1 + z_1 = 0$ (1), $w_2 + x_2 = 0, y_2 + z_2 = 0$ (2).

The vector $u + v = (w_1 + w_2, x_1 + x_2, y_1 + y_2, z_1 + z_2) \in U$ if it satisfies the constraint on U that

$(w_1 + w_2) + (x_1 + x_2) = 0, (y_1 + y_2) + (z_1 + z_2) = 0$. Adding constraints (1) and (2) gives

$(w_1 + w_2) + (x_1 + x_2) = 0, (y_1 + y_2) + (z_1 + z_2) = 0$. Hence $u + v \in U$ and U is closed under addition. For any $a \in \mathbb{R}$, the vector $au = (aw_1, ax_1, ay_1, az_1) \in U$ if it satisfies the constraint

on U that $(aw_1) + (ax_1) = 0, (ay_1) + (az_1) = 0$. Multiplying constraint (1) by a gives

$(aw_1) + (ax_1) = 0, (ay_1) + (az_1) = 0$. Hence $au \in U$ and U is closed under scalar

multiplication. Therefore, U is a vector subspace of $\mathbb{R}^4$.

(d) No, because $x = \pm y$ is a nonlinear constraint equivalent to $x^2 = y^2$. The solution set of this equation is cross-shaped. Alternatively, $\pm$ means $+$ or $-$, so

$U = \{(x, y) \in \mathbb{R}^2 \mid x = y \text{ or } x = -y\} = \{(x, y) \in \mathbb{R}^2 \mid x = y\} \cup \{(x, y) \in \mathbb{R}^2 \mid x = -y\}$ and in general a union of vector subspaces is not a vector subspace.

$(0, 0)$ satisfies the constraint because $0 = \pm 0$, so U contains the zero vector. But a counterexample shows that U is not closed under addition: e.g. $(1, 1), (1, -1) \in U$ but $(1, 1) + (1, -1) = (2, 0) \notin U$ since $2 \neq \pm 0$.

(e) No, because the constraint is nonlinear. In particular, $(0, 0) \notin U$ since $0 - 0 \neq 1$, so U does not contain the zero vector.

(f) Yes, because the constraints, which could also be written as $x = 0, y = 0$, are homogeneous linear. In fact, $U = \{(0, 0) \in \mathbb{R}^2\}$, which is the zero vector space as a vector subspace of $\mathbb{R}^2$. A more formal proof that U is a vector subspace could easily be written as in the examples above.

2. [20] Let V be a vector space over a field $\mathbb{K}$ and let u and v be fixed elements of V . Consider the set $U = \{a u + b v \mid a, b \in \mathbb{K}\}$. Show that (a) U is a subset of V and (b) U is a vector subspace of V . List the elements of U when $\mathbb{K} = \mathbb{F}_2$.

Solution

(a) By the vector space closure axioms, $a u + b v \in V$. Hence $U \subseteq V$.

(b) Taking $a = 0, b = 0$ implies $0_V \in U$ so U contains the zero vector. If $a_1, a_2, b_1, b_2 \in \mathbb{K}$ then $a_1 u + b_1 v, a_2 u + b_2 v \in U$ and $(a_1 u + b_1 v) + (a_2 u + b_2 v) = (a_1 + a_2) u + (b_1 + b_2) v \in U$ since it has the required form because $(a_1 + a_2), (b_1 + b_2) \in \mathbb{K}$, so U is closed under vector addition. If $k \in \mathbb{K}$ then $k(a u + b v) = (k a) u + (k b) v \in U$ since it has the required form because $k a, k b \in \mathbb{K}$, so U is closed under scalar multiplication. Therefore, U is a vector subspace of V .

When $\mathbb{K} = \mathbb{F}_2 = \{0, 1\}$, enumerating all possible values of a and b shows that $U = \{0, u, v, u + v\}$.

3. [20] Let $V = M_{2,2}(\mathbb{R})$ be the vector space of real 2×2 matrices with the usual addition and multiplication by a scalar. Let

$$U_1 = \left\{ \begin{bmatrix} a & b \\ c & a \end{bmatrix} \mid a, b, c \in \mathbb{R} \right\}, \quad U_2 = \left\{ \begin{bmatrix} d & 0 \\ 0 & -d \end{bmatrix} \mid d \in \mathbb{R} \right\}.$$

Prove that U_1 and U_2 are vector subspaces of V . What would you guess their dimensions to be and why? [We will see how to determine dimension formally in Chapter 4.]

Solution

U_1 : Choosing $a = b = c = 0$ shows that the zero matrix is an element of U_1 . Consider two elements of U_1 with matrix elements denoted by subscripts 1 and 2. Then

$$\begin{bmatrix} a_1 & b_1 \\ c_1 & a_1 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 \\ c_2 & a_2 \end{bmatrix} = \begin{bmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & a_1 + a_2 \end{bmatrix}$$

and for all $k \in \mathbb{R}$

$$k \cdot \begin{bmatrix} a & b \\ c & a \end{bmatrix} = \begin{bmatrix} k a & k b \\ k c & k a \end{bmatrix}$$

Both right hand sides have the required form to be elements of U_1 , so U_1 is closed under vector space operations. Hence U_1 is a vector subspace of V . The general element of U_1 contains 3 parameters, so guess that U_1 has dimension 3.

U_2 : Choosing $d = 0$ shows that the zero matrix is an element of U_2 . Consider two elements of U_2 with matrix elements denoted by subscripts 1 and 2. Then

$$\begin{bmatrix} d_1 & 0 \\ 0 & -d_1 \end{bmatrix} + \begin{bmatrix} d_2 & 0 \\ 0 & -d_2 \end{bmatrix} = \begin{bmatrix} d_1 + d_2 & 0 \\ 0 & -d_1 - d_2 \end{bmatrix}$$

and for all $k \in \mathbb{R}$

$$k \cdot \begin{bmatrix} d & 0 \\ 0 & -d \end{bmatrix} = \begin{bmatrix} kd & 0 \\ 0 & -kd \end{bmatrix}$$

Both right hand sides have the required form to be elements of U_2 , so U_2 is closed under vector space operations. Hence U_2 is a vector subspace of V . The general element of U_2 contains 1 parameter, so guess that U_2 has dimension 1.

4. [30] Let $S = \{(x, y, z) \in \mathbb{R}^3 \mid x = 2y\}$, $T = \{(x, y, z) \in \mathbb{R}^3 \mid y = -z\}$ be vector subspaces of the vector space $\mathbb{R}^3$.

(a) Show that $S \cap T = \{(x, y, z) \in \mathbb{R}^3 \mid x = 2y, y = -z\}$ and hence prove that $S \cap T$ is a vector subspace of the vector space $\mathbb{R}^3$.

(b) Find a counterexample that proves that $S \cup T$ is not a vector subspace of the vector space $\mathbb{R}^3$.

(c) Show that S and T can also be written as

$S = \{(2y, y, z) \in \mathbb{R}^3 \mid y, z \in \mathbb{R}\}$, $T = \{(x, -z', z') \in \mathbb{R}^3 \mid x, z' \in \mathbb{R}\}$, where z' is used in T to show that is is not related to the z used in S . Hence, express $S + T$ in a similar form, namely

$\{(a, b, c) \in \mathbb{R}^3 \mid y, z, x, z' \in \mathbb{R}\}$, where you must find expressions for a, b, c . By finding values of y, z, x, z' such that $(a, b, c) = (X, Y, Z)$ for an arbitrary vector $(X, Y, Z) \in \mathbb{R}^3$, prove that $S + T = \mathbb{R}^3$ and is therefore a vector subspace of the vector space $\mathbb{R}^3$.

▼ Solution

(a) The intersection of S and T is the set of all elements that satisfy both the constraint on S and the constraint on T , so the result for $S \cap T$ follows; the comma separating multiple constraints can be interpreted as a logical "and".

Let $U = \{(x, y, z) \in \mathbb{R}^3 \mid x = 2y, y = -z\}$. Prove that U is a vector subspace of $\mathbb{R}^3$.

$(0, 0, 0) \in U$ since $0 = 2 \cdot 0$, $0 = -0$. If $(x_1, y_1, z_1), (x_2, y_2, z_2) \in U$ then $x_1 = 2y_1$, $y_1 = -z_1$ and $x_2 = 2y_2$, $y_2 = -z_2$. Adding the constraints shows that

$(x_1 + x_2) = 2(y_1 + y_2)$, $(y_1 + y_2) = -(z_1 + z_2)$, so $(x_1 + x_2, y_1 + y_2, z_1 + z_2) \in U$ since it satisfies the constraints on U .

(b) $(2, 1, 0) \in S$ and $(0, -1, 1) \in T$, so $(2, 1, 0), (0, -1, 1) \in S \cup T$, but $(2, 1, 0) + (0, -1, 1) = (2, 0, 1) \notin S \cup T$, since $(2, 0, 1) \notin S$ and $(2, 0, 1) \notin T$. Therefore $S \cup T$ is not closed under addition and so is not a vector subspace.

(c) Substituting the constraints into the general vectors to eliminate one of the parameters gives the forms shown. Note that the parameters can be renamed without changing the definitions of the subsets. Then the definition of the sum of two vector subspaces leads immediately to $S + T = \{(2y + x, y - z', z + z') \in \mathbb{R}^3 \mid y, z, x, z' \in \mathbb{R}\}$. Now set $(2y + x, y - z', z + z') = (X, Y, Z)$ and solve for y, z, x in terms of X, Y, Z, z' to give $x = X - 2(Y + z')$, $y = Y + z'$, $z = Z - z'$. This means that for any value of z' we can find values for y, z, x and hence every vector $(X, Y, Z) \in \mathbb{R}^3$ is also in $S + T$; therefore $S + T = \mathbb{R}^3$. (More precisely, $S \subseteq \mathbb{R}^3$ and $T \subseteq \mathbb{R}^3$ by definition so $S + T \subseteq \mathbb{R}^3$, and we have just shown that every vector in $\mathbb{R}^3$ is also in $S + T$ so $\mathbb{R}^3 \subseteq S + T$, therefore $S + T = \mathbb{R}^3$.) This analysis looks forward to the definition of spanning sets next week.